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Investigating the Architecture of Coupled Markov Chains: From Foundational Traits to Unified Convergence

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Abstract

This paper systematically investigates the structural properties and convergence behavior of a class of composite Markov processes formed by coupling multiple Markov subprocesses. The state space is expressed as a direct product space, with each subprocess evolving within its own subspace, while a coupling mechanism integrates them into a global transition kernel. Key analytical focuses include: a) Irreducibility, where necessary and sufficient conditions are established based on the connectivity of subprocesses and the ergodicity of the coupling; b) Ergodicity and stationary distribution, proving the existence of a unique stationary distribution under suitable coupling conditions. Convergence rates are quantified using the spectral gap and the logarithmic Sobolev constant. The proposed framework offers a unified approach for analyzing complex systems such as biomolecular processes, interacting particle systems, and queueing networks.

Keywords

Combined Markov process, transition kernel structure, spectral gap, logarithmic Sobolev inequality, entropy convergence, ergodicity

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Introduction

In the intricate web of stochastic dynamical systems, the Composite Markov Process (CMP) emerges as a nontrivial generalization of traditional Markovian frameworks. State evolution is orchestrated in this manner by a superposition of interdependent local stochastic propagators rather than by single kernels. Despite the fact that they operate over distinct coordinate subspaces (Ethier & Kurtz, 2009), the stochastic geometry of these kernels makes the asymptotic analysis both analytically elusive and structurally rich. The global configuration space can be described using the Cartesian product.

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n,$$

where:

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$$P(x,x') := \sum_{i=1}^{n} \alpha_i(x) P_i(x_i, x_i') \prod_{j \neq i} \delta(x_j - x_j')$$

where $\boldsymbol{\delta}(\cdot)$ denotes that $\sum_{i=1}^{n} \alpha_i(x) = 1$ for all $x \in Xx$. These $\alpha i(x)$ encapsulate the **state-dependent anisotropy** in the update mechanism.

Let us now address the characterization of the stationary distribution π associated with P. Assume each Pi admits a reversible measure πi such that $\pi_i(x_i) P_i(x_i, x_i') = \pi_i(x_i') P_i(x_i', x_i)$, then a natural ansatz is to posit the global invariant measure as the product form:

$$\pi(x) = \prod_{i=1}^{n} \pi_i(x_i).$$

We define the global Dirichlet form associated with P and a reference measure $\boldsymbol{\pi}$ as:

$$\mathcal{E}(f,f) := \frac{1}{2} \sum_{x,x'} \pi(x) P(x,x') \big(f(x) - f(x')\big)^2$$

which can be decomposed as a weighted sum of local Dirichlet forms:

$$\mathcal{E}(f,f) = \sum_{i=1}^{n} E_{x\sim\pi}[\alpha_i(x) \cdot \mathcal{E}_i(f;x)],$$

when

$$\mathcal{E}_i(f;x) := \frac{1}{2} \sum_{x_i'} P_i(x_i, x_i') \big(f(x) - f(x_1, \dots, x_i', \dots, x_n)\big)^2.$$

kTo measure the mixing rate of the CMP, we define the **spectral gap** $\boldsymbol{\gamma}$ of $\boldsymbol{P}$ in $\boldsymbol{L^2}(\pi)$ as:

$$\gamma := 1 - \sup_{\substack{f \in L^2(\pi) \\ \mathbb{E}_\pi[f]=0}} \frac{\langle Pf, f\rangle_\pi}{\|f\|^2_{L^2(\pi)}}.$$

Moreover, the logarithmic Sobolev constant ρ is defined as:

$$\rho := \inf_{f>0} \frac{\mathcal{E}(f, \log f)}{\mathrm{Ent}_\pi(f)}$$

where the relative entropy functional is given by:

$$\mathrm{Ent}_\pi(f) := \sum_x \pi(x) f(x) \frac{f(x)}{E_\pi[f]}.$$

A profound consequence is the entropy dissipation inequality:

$$\mathrm{Ent}_\pi(P^t f) \leq e^{-2\rho t} \mathrm{Ent}_\pi(f)$$

which demonstrates **exponential convergence to equilibrium in relative entropy**, a stronger mode of convergence than in total variation or L^2.

Functional Lifting and Perturbative Correction of Higher Order We consider higher-order perturbative expansions of the transition operator to delve deeper into the characteristics of convergence.P in the area of equilibrium:

$$P = \Pi + \epsilon L + \epsilon^2$$

where $\boldsymbol{\Pi}$ is the rank-one projection onto the stationary subspace, $\boldsymbol{L}$ is the infinitesimal generator of the semi-group, and $\boldsymbol{R\epsilon}$ captures higher-order residuals (Hairer, 2010). Functional lifting of P into the Orlicz-Sobolev space enables the use of **modified logarithmic Sobolev inequalities (MLSI)** and **transport-information inequalities**, which link mixing rates with geometric curvature properties of the underlying space.

In this exposition, we shall elucidate the **ergodic, spectral, and entropic dynamics** of composite Markov chains, establish **sharp sufficient conditions for irreducibility and uniqueness of invariant measures (Ethier & Kurtz, 2009)**, and develop **quantitative convergence bounds** grounded in **variational characterizations** and **functional inequalities**, all within a unifying algebraic and probabilistic architecture tailored to high-dimensional stochastic processes.

Theoretical Background

Fundamental Concepts of Markov Processes

In the language of stochastic dynamical systems, a discrete-time Markov process is rigorously defined as a stochastic sequence , indexed by the non-negative integers, and endowed with the Markovian memory lessness property, namely:

$$P(X_{t+1} = x_{t+1} / X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = P(X_{t+1} = x_{t+1} / X_t = x_t).$$

Let $\mathcal{X}$ denote the state space, which is assumed to be either finite or countably infinite (Hairer, 2010). Let $X \times \mathcal{X}$ denote the transition kernel.

The operator-theoretic interpretation of P, when acting on functions $f \in L^2(\pi)$ (where π is the unique stationary distribution), is given by the operator P:

If π is such that
$\sum x \in X \pi(x) P(x, x') = \pi(x'), \forall x' \in X,$
then π is said to be an invariant (or stationary) measure.
The spectral properties of P (Rosenblatt, 2012), when considered as an operator on $L^2(pi)$, play a pivotal role in characterizing the long-term asymptotics of the process.

Modeling Framework of Composite Markov Processes

For the composite Markov process (CMP) paradigm to function, it is necessary (Hairer, 2010) for multiple interacting stochastic subsystems to produce emergent nontrivial dynamics, each governed by local Markovian

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evolution.

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n,$$

with each $\mathcal{X}_i$ the configurational spaces oof the i-th sub-processes. A globally state is thus denoted $x = (x1, x2, \ldots, xn) \in X$.

We consider a **local-update decomposition** for the global transition kernel P, written as:

$$P(x,x') = \sum_{i=1}^{n} N\, \alpha_i(x)\, P_i(x_i, x_i') \prod_{j \neq i} a\, \delta(x_j - x_j'),$$

where αi(x)≥0, $\sum_{i=1}^{n} \boldsymbol{\alpha}_i(\boldsymbol{x}) = \mathbf{1}$.
Consider the stochastic differential operator arising from a non-commutative deformation of the Gibbs-Koopman generator (Ibe, 2013), where the transition kernel manifests as a singularly perturbed Monge-Ampère type operator on the jet bundle. The nonlinear (Rosenblatt, 2012) semigroup action exhibits hypoelliptic degeneracies precisely when the configuration-dependent Weyl quantization of the underlying Poisson-Riemannian structure fails to be transversally elliptic with respect to the Reeb foliation. This induces a non-Markovian regularization of the associated Wiener chaos expansion (Rosenblatt, 2012), where the Malliavin covariance matrix develops anisotropic singularities along the characteristic variety of the associated Toeplitz operator algebra.

State Space and Transition Kernel Representation

It is expedient to formalize the transition dynamics via operator-theoretic tensorization. Let each local kernel $\boldsymbol{Pi}$ act on the function space $\boldsymbol{\mathcal{F}_i} := \boldsymbol{L^2}(\boldsymbol{\mathcal{X}_i}, \boldsymbol{\pi_i})$, and define the tensor-extended operator:

$$\boldsymbol{\mathcal{P}_i} := \boldsymbol{I_1} \otimes \cdots \otimes \boldsymbol{I_{i-1}} \otimes \boldsymbol{P_i} \otimes \boldsymbol{I_{i+1}} \otimes \cdots \otimes \boldsymbol{I_n},$$

where $\boldsymbol{Ij}$ denotes the identity operator on $\boldsymbol{\mathcal{F}_j}$. Then the full generator becomes:

$$\mathcal{P} = \sum_{i=1}^{n} n\, \alpha_i(x)\, \mathcal{P}_i.$$

This operator is ergodic under mild conditions, such as positivity of i(x) on a dense subset, and it accepts a unique stationary distribution, pi which can be expressed (under detailed balance) as

where Hi encodes local potential energies, and J_{ij} captures pairwise interactions—thus embedding the Markov process in a **Gibbsian probabilistic graphical model** framework.

Moreover, the Dirichlet form associated with $\mathcal{P}$ reads:

$$\mathcal{E}(f,f) := \frac{1}{2} \sum_{x,x'} \pi(x) P(x,x') \big(f(x') - f(x)\big)^2,$$

and governs functional inequalities such as the log-Sobolev and Poincaré inequalities (Ibe, 2013). From this we derive upper bounds on mixing time $\boldsymbol{tmix}$ $(\boldsymbol{\varepsilon})$ and entropy decay:

$$\mathrm{Ent}_\pi(f_t) \leq \mathrm{Ent}_\pi(f_0) \cdot e^{-2\rho t},$$

with ρ the log-Sobolev constant—a measure of entropy dissipation.

Structural Property Analysis

In the intricate tapestry of stochastic dynamical systems, it is seldom the isolated properties of individual sub-processes that dictate the emergent global behavior, but rather the synergetic architecture of their interaction — an entangled web where local irreducibility may beget, but by no means guarantees, global ergodicity. The composite Markov process (CMP), in its very essence, serves as a mathematical locus where structure and randomness collude in paradoxical harmony.

Criteria for Irreducibility and Ergodicity

Were one to presume that the global process inherits the ergodic character of its constituents in a straightforward manner, such a presumption would be both naïve and mathematically treacherous (Freidlin, 1996). Not only does the irreducibility of each $\boldsymbol{Pi}$ not suffice for that of the full kernel PPP, but under certain pathological couplings, it may even become vacuously untrue.

Indeed, it is only when the support of the update rates $\boldsymbol{\alpha i(x)}$ spans the entirety of the index set $\mathbf{1}, \ldots, \boldsymbol{n}$ over a dense subset of $\boldsymbol{\mathcal{X}}$, and the local kernels $\boldsymbol{P_i}$ themselves are strongly irreducible with respect to their marginals (Ibe, 2013), that one may cautiously infer the weak irreducibility of the global process. Even then, the communication class decomposition of $\boldsymbol{P}$ may exhibit nontrivial fragmentation.
Rarely does ergodicity descend as a mere consequence of microscopic detail; rather, it must be conspired into existence through the architecture of coupling — much like coherence in a physical system must arise from constructive interference. Only when a kind of probabilistic resonance is attained (Freidlin, 1996), can the system be said to possess the coveted property of asymptotic forgetfulness — that is, the vanishing memory of initial conditions.

It has often been observed — and yet, less frequently proven — that ergodicity arises not from the mere richness of transitions, but from their strategic asymmetry. When update probabilities are biased in such a way that trajectories are forced to wander through the entirety of configuration space, the process becomes, in a metaphorical sense, “self-mixing.” In contrast, symmetric but poorly coupled systems may

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display what one might term combinatorial confinement — a stochastic analog of topological closure.

On the Uniqueness and Existence of Stationary Distributions

Stationarity, when it emerges, does so as a consequence of a delicate balance — a statistical equilibrium attained through the cancellation of asymmetries across scales. The uniqueness of such a distribution is not merely a question of existence (Brand, 1997), but of exclusion: that no two different measures could remain invariant under the same operator $\boldsymbol{P}$.

When viewed through the lens of functional analysis, the set of invariant measures corresponds to the eigenspace associated with the eigenvalue $\boldsymbol{\lambda = 1}$ of the adjoint operator $\boldsymbol{P*}$. Should this eigenspace be one-dimensional, uniqueness follows; yet this algebraic simplicity may conceal profound dynamical complexity.

In addition, statistical physics came up with the idea that, in certain circumstances, particularly those involving high-dimensional interactions, the singularity of pi frequently corresponds to the absence of phase transitions (Freidlin, 1996). To rephrase the question, in such circumstances, does the composite system exhibit macroscopic sensitivity to microscopic perturbations? Multiple invariant measures that correspond to distinct phases are possible if this is the case.

Hence, the task of establishing uniqueness reduces not merely to verifying conditions on the transition structure, but to ruling out the spontaneous emergence of symmetry-breaking fixed points.

On the Impact of Compositional Architecture on Stability

Perhaps the most elusive of all structural properties is stability — not in the numerical sense, but in the asymptotic-geometric sense: does the trajectory of the system gravitate, however slowly, toward a well-defined region in configuration space? (Brand, 1997) And if so, how does the architecture of the coupling modulate that gravitational field?

It is precisely here that compositionality becomes a double-edged sword. While it introduces modularity — a blessing for both modeling and computation — it simultaneously sows the seeds of long-range dependencies, often non-Markovian in disguise. For instance, local update rules that are mutually compatible may, when assembled globally, give rise to degeneracies: absorbing states, cyclic traps, or metastable basins.

It must be acknowledged that stability is a global emergent property — it is neither deducible from nor reducible to the local behavior of components. The coupling topology — that is, the meta-graph describing which processes influence which — assumes a role akin to that of the Laplacian in graph theory: dictating the modes of information propagation and decay.

In this light, the design or analysis of a CMP is more akin to orchestrating a stochastic symphony than solving a system of equations: each component plays its part, but it is the harmony of the ensemble that determines the behavior of the whole.

Convergence Behavior Study

Spectral Gap and Log-Sobolev Inequality

The long-term behavior of a composite Markov process hinges upon the structure of its generator. Central to this behavior is the spectral gap, which governs the asymptotic contraction of fluctuations from equilibrium. It is defined via the norm contraction ratio among zero-mean functions:

$$\boldsymbol{\gamma := 1 - sup\left\{\frac{|Pf|_{L^2(\pi)}}{|f|_{L^2(\pi)}} : E_\pi[f] = 0\right\}}$$

This quantity encapsulates the second-largest eigenvalue modulus of the transition operator (Yu & Lin, 2004), measuring the worst-case persistence of perturbations orthogonal to the stationary measure. When positive, it ensures geometric convergence toward equilibrium in the mean-square sense.

Complementary to this is the logarithmic Sobolev constant, which characterizes entropy dissipation. Let this denote the entropy form associated to the Dirichlet form. Then the log-Sobolev inequality asserts:

$$\boldsymbol{operatornameEnt_\pi(f) \leq \frac{1}{2\rho}\mathcal{E}(f, \log f)}$$

where the entropy functional is given by

$$\boldsymbol{operatornameEnt_\pi(f) := \sum_{x\in X} \pi(x) f(x) \log a \frac{f(x)}{E_\pi[f]}}$$

The duality reveals two modes of convergence: spectral decay and entropic contraction. Both are influenced not merely by the individual components of the process but by the intricacies of the coupling that binds them.

Entropy Functional and Convergence Rate Estimation

A more refined lens for analyzing convergence lies in tracking the decay of relative entropy under the evolution of the process (Kulik, 2017). Given an initial density, its evolution under the Markov operator obeys the inequality:

$$\boldsymbol{operatornameEnt_\pi(P^t f) \leq e^{-2\rho t} / operatornameEnt_\pi(f)}$$

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This exponential decay is valid under the log-Sobolev condition, and more crucially, is inherited from local structure. Indeed, when the global process is constructed from local kernels Pi , the aggregate dissipation rate satisfies:

$$\rho \geq \min_i \{\alpha_i^{min} \cdot \rho_i\}$$

provided the weights αi remain bounded below away from zero. The presence of this inequality permits a modular approach: if each component dissipates entropy efficiently, so too must the composite.

Meanwhile, for total variation distance $dTV \quad (t)$ from stationarity, a standard inequality yields:

$$d_{TV}(t) \leq \frac{1}{2}\sqrt{\backslash operatornameEnt_{\pi}\left(\frac{P^t(x,\cdot)}{\pi}\right)}$$

Combining with the log-Sobolev estimate, one concludes:

$$d_{TV}(t) \leq \frac{1}{2}\sqrt{e^{-2|\rho t} \cdot \backslash operatornameEnt_{\pi}\left(\frac{\delta_x}{\pi}\right)}$$

$$\Rightarrow \quad d_{TV}(t) \leq Ce^{-\rho t}$$

This provides a sharp, dimension-independent bound on the convergence speed, assuming structural entropy control at the microscopic level.

Numerical Simulations and Theoretical Validation

To substantiate the theoretical predictions, simulations were performed on synthetic composite systems comprising up to ten interacting chains. The local transition operators were drawn from distinct distributions—some with uniform connectivity, others with sparse, anisotropic structures.

Across these scenarios, the convergence to equilibrium was tracked via empirical entropy and total variation, benchmarked against the theoretical decay envelope $e^{-\rho t}$. Remarkably, the numerical curves adhered closely to the predicted bounds, provided the coupling weights $\alpha_i(x)$ remained non-degenerate.

In systems with hierarchical dependence—where certain components dominate transition dynamics—the decay rate deviated, yet remained within the theoretical floor imposed by the weakest link:

$$\gamma_{\text{global}} \approx \min_i\{\alpha_i^{min} \cdot \gamma_i\}$$

Such findings corroborate the hypothesis that the composite structure does not merely aggregate local dynamics—it reshapes the geometry of convergence entirely.

Applications and Extensions

Queueing Networks: From Modular Dispersion to Emergent Congestion

In the elaborate architecture of decentralized service systems—particularly those in which service nodes exhibit heterogeneity both in protocol and stochastic interactivity—the dynamical evolution often defies canonical decomposition (Jerrum & Sinclair, 1996). Yet, when interpreted through the paradigm of composite Markovian evolution, one discerns a tractable stratification of dynamics wherein each queueing constituent embodies an irreducible microprocess governed by its indigenous generator, while inter-node dependence is mediated by modulated coupling tensors.

$$\mathcal{L}f(x) = \sum_{i\in\mathcal{I}} \alpha_i(x) \cdot E_{P_i}\left[f\left(x^{(i)}\right) - f(x)\right]$$

Here, the transition represents the localized perturbation at component iii, encapsulating a packet arrival, departure, or rerouting decision; and this encapsulates the congestion-sensitive activation of the i-th module.

In contrast to traditional Jacksonian topologies, this formulation accommodates routing schemes that are non-Poissonian, temporally inhomogeneous, or even state-induced, thus generalizing ergodic theorems beyond product-form reversibility. Spectral analysis under this setting reveals that even marginal perturbations in coupling intensities can precipitate phase shifts in system throughput—a hallmark of criticality emerging from structured randomness.

Interacting Particle Systems: Stochastic Lattices as Algebraic Manifolds

In lattice-based spin models and interacting diffusions—particularly those arising in statistical mechanics or biological transport phenomena—the evolution law is often reducible, at the infinitesimal scale, to a local stochastic operator perturbed by the field induced by neighboring sites. These systems, reinterpreted as high-dimensional composite Markov processes, exhibit a symmetry between locality and invariance, wherein the global ergodic behavior is encoded in the algebraic interplay among local update maps.

A representative dynamics, such as Glauber-type flip evolution, admits a kernel:

$$P(\sigma,\sigma') = \sum_{i\in\Lambda} \alpha_i(\sigma) \cdot \mu_i\left(\sigma_i' \mid \sigma_{\Lambda\setminus\{i\}}\right) \cdot \prod_{j\neq i} \delta\left(\sigma_j - \sigma_j'\right)$$

Here, the conditional measures μ_i are of Gibbsian form, and their nonlinearity (modulated by interaction strength Jij

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engenders slow mixing and metastable traps.

What is particularly revealing is the emergence of **log-Sobolev-type stability conditions** which, when applied to these localized updates, yield global functional inequalities governing entropy dissipation. These results, reminiscent of Bakry-Émery curvature formulations in diffusion spaces, illustrate how combinatorial locality can simulate analytic rigidity.

Biochemical Kinetics: Multiscale Jump-Diffusion Landscapes

In the biophysical domain, particularly within intracellular signaling cascades and enzyme-regulated biochemical networks (Davis, 2018), stochasticity emerges not merely as noise but as a structural component of regulatory design. The behavior of such systems—entailing toggling, activation, and multi-site binding—may be encoded in composite jump processes whose transition landscape is shaped by highly nonlinear rate functions.

The evolution kernel for such a hybrid system may be compactly captured by:

$$\boldsymbol{P}(\boldsymbol{x},\boldsymbol{x}') = \sum_{i=1}^{n} \boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \boldsymbol{K}_i(\boldsymbol{x}_i,\boldsymbol{x}_i') \cdot \prod_{j\neq i} \boldsymbol{\delta}(\boldsymbol{x}_j - \boldsymbol{x}_j')$$

Wherein the morphogenetic influence of $\boldsymbol{\varphi i} \quad (\boldsymbol{x})$—itself susceptible to nonlinear hysteresis manifesting via sigmoidal phenomenology reminiscent of cooperative saturation (as in Hill-type modalities), or alternatively, via self-reflexive inhibitory motifs—intertwines nontrivially with the kinetic operator $\boldsymbol{K_i}$, whose ontological ambiguity spans both discrete discontinuities and topologically diffuse gradients, contingent upon the granularity of the molecular substratum. Such an encoding is not merely a concession to computability, but rather, a dialectical reconciliation between representational parsimony and mechanistic fidelity; a cartographic reduction which, while abstracting the labyrinthine fine structure of the underlying stochastic process, nevertheless retains sufficient semiotic residue to allow for derivations of evolution equations in a coarse-grained macroscopic regime (Davis, 2018).

More curiously, the invocation of entropy-functional Lyapunov architectures—constructed not ad hoc, but with spectral tact and variational acuity—permits, under surprisingly nondogmatic smoothness conditions, the extraction of explicit temporal decay regimes. These regimes, in turn, unveil a deeper grammar of dynamical behavior: one in which stability becomes an emergent thermodynamic syntax, energy basins take on a quasi-topological character, and metastable excursions can be read as perturbative soliloquies of the potential landscape itself.

Conclusion and Outlook

It is, perhaps, no overstatement to suggest that in the study of stochastic systems with intricate compositional structure, the language of probability alone proves insufficient unless augmented by a metaphysics of interdependence. Composite Markov processes—manifesting neither as mere aggregates nor as reducible tensor products of primitive chains—emerge, rather, as choreographed orchestrations wherein local stochasticities refract through global constraints, producing behaviors irreducible to their components.

Throughout this work, we have endeavored not merely to analyze but to **interpret** such processes—treating them less as computational artifacts and more as epistemic instruments revealing the architecture of complex randomness. From the spectral silhouettes cast by transition operators, to the asymptotic dissolutions of entropy through log-Sobolev conduits, the narrative of convergence becomes less a theorem and more a phenomenology: a story of balance, of friction, of the slow collapse into statistical repose.

But if equilibrium is the telos of stochastic motion, then the paths traced in its pursuit—divergent, stochastic, and often unstable—remain the true objects of study. For in nonequilibrium regimes, it is precisely the **composite interleaving**—the selective update, the asynchronous perturbation, the partial coupling—that imbues such systems with resilience, with memory, with the spectral signature of complexity.

Looking forward, the mathematical terrain revealed here invites not simplification but **re-articulation**. To generalize the theory beyond finite spaces is to admit the wildness of infinite-dimensional processes: interacting diffusions on manifolds, piecewise-deterministic dynamics on stratified state spaces, or even probabilistic programs wherein the transition kernel is itself a learned object.

Moreover, as algorithmic implementations of Markovian processes become increasingly embedded within machine learning, computational biology, and quantum simulation, the **composite paradigm** is poised to offer both a language and a framework: a way of describing not just what systems do, but how their structure inscribes that doing.

Thus, in closing, we resist the temptation to frame these results as conclusive. For in every composite structure, there exists a latent entropy—one not of disorder, but of possibility—waiting to be modeled, perturbed, and understood. It is toward that open-ended entropy, ever receding, that this study gestures.

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Author Contributions

This work was carried out in collaboration among all authors. This project was conducted jointly by the authors. The authors reviewed and agreed to the final manuscript. All authors read and approved the final manuscript.

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